2018 IEEE/ACM International Conference on Advances in Social Networks Analysis and Mining (ASONAM)

Identifying Influential Nodes to Inhibit Bootstrap Percolation on Hyperbolic Networks

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***Abstract*—This work involves agent-based simulation of bootstrap percolation on hyperbolic networks. Our goal is to identify influential nodes in a network which might inhibit the percolation process. Our motivation, given a small scale random seeding of an activity in a network, is to identify the most influential nodes in a network to inhibit the spread of an activity amongst the general population of agents. This might model obstructing the spread of fake news in an on line social network, or cascades of panic selling in a network of mutual funds, based on rumour propagation. Hyperbolic networks typically display power law degree distribution, high clustering and skewed centrality distributions. We introduce a form of immunity into the networks, targeting nodes of high centrality and low clustering to be immune to the percolation process, then comparing outcomes with standard bootstrap percolation and with random selection of immune nodes. We generally observe that targeting nodes of high degree has a delaying effect on percolation but, for our chosen graph centralisation measures, a high degree of skew in the distribution of local node centrality values bears some correlation with an increased inhibitory impact on percolation.**

***Index Terms*—Hyperbolic Random Geometric Graphs; Bootstrap Percolation; Inhibitor Nodes; Influential Nodes**

I. Introduction

In real world complex networks, there is a lot of interest in determining influential nodes in networks, for example identifying social media influencers to broaden the reach of brand messaging. This work involves identifying nodes in a network which are influential in obstructing the spread of an activity in bootstrap percolation on hyperbolic networks. Measures of network influence are often based on centrality measures such as degree, betweenness and closeness centrality. Degree centrality is frequently chosen, as this value is determined by the number of links incident to a node, and high degree represents a node with many contacts. High betweenness centrality represents nodes which may have a "brokerage" role in a network. High closeness centrality represents nodes that are closely linked to important nodes while not necessarily having high degree or betweenness. Our approach is aiming for an a priori intelligent selection of influence by investigating any correlation between graph properties that are highly skewed in the hyperbolic graphs such as high closeness, and high clustering. Our intuition is that properties with highly skewed distributions will maximise the inhibitory effect.

IEEE/ACM ASONAM 2018, August 28-31, 2018, Barcelona, Spain
978-1-5386-6051-5/18/$31.00 © 2018 IEEE

In bootstrap percolation, people are often interested in determining which nodes to choose, or how many, to maximise the spread of an activity, for example who to target in viral marketing to optimise the spread. Our own focus is on inhibiting this spread, and observing which nodes are most influential in obstructing the activity; there has been little empirical investigation of this nature on hyperbolic networks, which share many features with real world complex networks. This might model the spread of fake news in a social network, where we wish to identify "responsible" nodes, which will take care not to share disreputable information, or perhaps a network of mutual funds, where there is panic selling based on rumour, certain influential nodes could be targeted to hold firm against the spread of fake information. This work is related to the study of immunisation strategies in epidemiology, and other inhibitory actions in networks, such as preventing the spread of malware in computer networks. While all are interested in identifying the best nodes to inhibit the spread of an activity, there are different underlying assumptions. In epidemiology and the study of computer viruses, the specific infection must be identified in order to introduce immunity by applying a vaccine or anti-viral software. In bootstrap percolation, modelling social influence in spreading activities, the message/ activity may never previously have been encountered; in our work, the inhibitor node receives the message, but resists peer pressure, and takes no part in spreading the activity.

Our work complements existing theoretical work on identifying influential nodes in networks and outlines further avenues in an empirical manner. Our proposed method for identifying these nodes involves agent based modelling of bootstrap percolation on hyperbolic random geometric graphs, to observe how the underlying spatial configuration of a network will impact upon the process. We have created a set of hyperbolic graphs of varying edge density, from unconnected to fully connected, and noted that as we increase the number of edges we can observe a distinct threshold for edge density above which the activity spreads and below which the activity fails to percolate. Our particular interest is on the set of graphs at the percolation threshold, to identify certain properties of these graphs that facilitate or impede percolation.

Section II provides related information about bootstrap percolation and hyperbolic graphs. Section III introduces our approach for identifying influential nodes to inhibit bootstrap percolation on hyperbolic networks. Section IV describes our experimental set-up and our results are presented in section V. Our conclusions are discussed in Section VI, together with suggestions for future work.

II. Background

A. Bootstrap Percolation

Bootstrap percolation describes the process where an activity spreads to individuals in a population when the number of their active contacts is at least a specified activation threshold. This concept was introduced in research studying the mechanism of ferro-magnetism [1].

This process can be applied as a model for social reinforcement, under the assumption that individuals are considered more likely to adopt a new activity that is popular with their contacts, and has been used to model a diverse variety of topics, such as the diffusion of information [2], [3], viral marketing [4], spreading behaviours [5], as observed in opinion formation, voter trends, the adoption of products, technology and social networking innovation [6]–[8], and in cascading power system failures.

In standard bootstrap percolation, an agent is in either active or inactive state. From an initial population of inactive agents, a set of active nodes is selected either at random or deterministically. An activation threshold is selected, and at each time step the activity spreads when the threshold of active contacts is exceeded. The activation mechanism occurs simultaneously at each time step for all agents in the population. The process repeats until either no further nodes can be activated, or until predefined criteria are met, for example the number of time steps. In the spatial form of the process, where agents are attached to nodes in a network, activation occurs where the number of active nodes directly connected to an inactive node exceeds the activation threshold. Bootstrap percolation has been studied on a number of random graphs and complex networks [9]–[14].

A key consideration in the field is whether or not an activity will spread completely across a network. Much of the research involved focuses on the relative state of the initial active seed set, and the emergent state of the population. This typically requires analysing which nodes are selected [15] for the initial seed set, or how many are required [16], to facilitate the spread of the activity. Kempe et al. [17] analysed algorithms to optimise the selection of influential nodes on a variety of networks, noting that this was NP-hard.

Candellero et al. [18] researched bootstrap percolation on hyperbolic random geometric graphs by analysing node position within the hyperbolic disc, whereby inductive reasoning was used to determine the size of the initial seed set for which the activity would percolate completely, fail to percolate, or spread to a positive amount with high probability.

B. Immunisation and Influential Nodes

In this context, the notion of immunity comes from epidemiology and the spread of diseases. In 2002 Pastor-Satorras and Vespignani introduced immunised nodes into a simple SIS (Susceptible - Infected - Susceptible) model of scale free networks such as the sexual partnership web and the Internet [19]. They noted that uniform random immunity had little effect, due to the inhomogeneity of node degree, and suggested that successful immunisation strategies should target immunity based on node degree ranking.

The topological differences in complex networks has meant that over the years a variety of measures have been used to identify influential nodes in networks. In a recent comparative review of progress in this area, Lu et al. clarified concepts and commonly used measures of influence, noting that several studies have found different measures to be influential based on whether the goal was to facilitate or obstruct the progress of the activity [20].

Recent work has looked at hybrid strategies, taking into account network structure and node activity [21]–[23]. Ghanbari et al have looked at correlation of centrality measures in cascade failures noting that, in their networks, removing nodes of high degree had less severe impact than removing nodes of lower degree [24].

Our own focus is based on a long term goal of being able to make an intelligent selection of influential nodes based on global network properties. With this in mind, we select hyperbolic random geometric graphs as our network model, as they display distinctive patterns of centrality measures as the number of edges in the graphs increases, which aids in mapping the relationship between the chosen node property and the inhibitory effect.

C. Hyperbolic Random Geometric Graphs

A random geometric graph is constructed by random selection of a number of points in some space of interest and then connecting those that lie within a specified distance of each other. They were developed to model real world applications, such as the likely spread of a forest fire given the distance between trees.

Random geometric graphs are typically created on the Euclidean plane, in particular the unit square and the unit disc. These graphs may be used to model wireless ad-hoc and sensor networks, in which node proximity defines network connections. Other applications for this model include percolation theory [18], diffusion within a network and conductivity [25].

Hyperbolic random geometric graphs were investigated by Krioukov et al in [26]. In this model, points are distributed within a hyperbolic disc of interest and pairs of nodes are connected if the hyperbolic distance between them is less than a specified distance parameter.

To demonstrate the difference between Euclidean and Hyperbolic random graphs, the negative curvature of the hyperbolic plane can be transformed to a 2 dimensional disc using the Poincaré disc model of the hyperbolic plane. The hyperbolic model effectively has more capacity than the Euclidean disc, with the circumference increasing exponentially as the radius grows larger. Points are not uniformly distributed within the hyperbolic disc, points in the centre appear closer to us, with outer points exponentially distant towards infinity at the boundary. This has the effect of creating graphs with lots of central hub nodes and leaf nodes towards the boundary.

Typically hyperbolic models are highly clustered [27], with power law degree distribution and short path lengths, attributes commonly seen in real world complex networks. Recent work suggests that the Internet graph conforms to an underlying hyperbolic geometry [26]. Figure 1 shows a snapshot of internet connectivity, produced by CAIDA, with selected Internet Service Providers coloured separately, illustrating the tree- like and highly clustered structure of the internet graph.

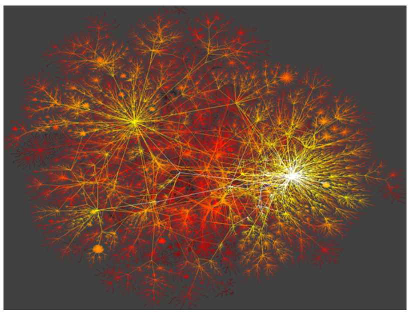


Fig. 1: The Topology of the Internet. Source: [28]

III. Framework for Identifying Influential Nodes to Inhibit Bootstrap Percolation on Hyperbolic Networks

In standard bootstrap percolation, the only possible state change is from inactive to active; no reverse state change is permissible. In a percolating network, our goal is to target the most influential nodes in obstructing the percolation process. Our motivation, given a small scale random attack in a network, is to select a small number of nodes to immunise against the activity, which would obstruct the percolation process and effectively grant a form of herd immunity.

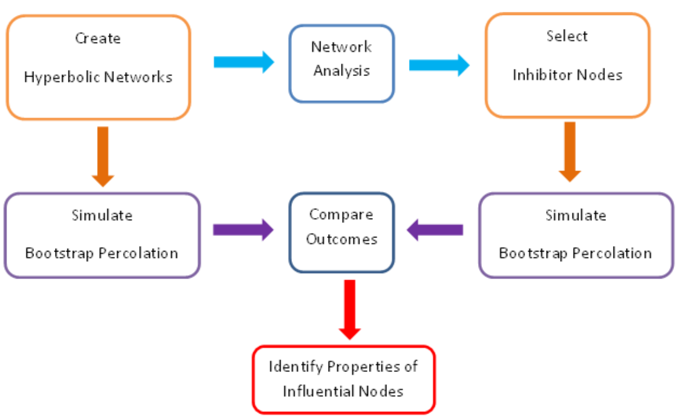


Fig. 2: Conceptual Framework for Identifying Influential Nodes to Inhibit Bootstrap Percolation on Hyperbolic Networkd

In our previous research [29], we found that as the number of edges in a set of hyperbolic graphs increased there was a distinct transition threshold above which the activity completely percolated on all the graphs and below which the activity failed to percolate on these graphs, illustrated in Figure 3. If a percentage of nodes was allowed to recover at each time step, this had a delaying effect on the percolation process and this effect was enhanced when nodes of high degree were selectively targeted to recover, compared with random selection of nodes.

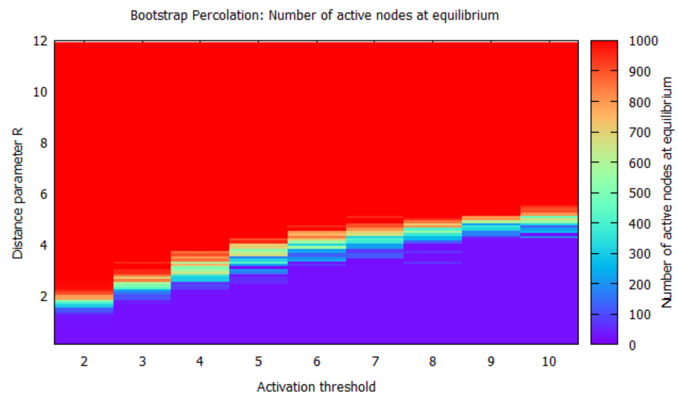


Fig. 3: Heat Map for $A_o = 20$, with AT from 2 to 10, and $R =$ 0.1 to 12, showing number of final active nodes at equilibrium

Our current focus is on selecting the most influential nodes in a network to obstruct the percolation process. Instead of a percentage of active nodes recovering at each time step, the targeted nodes would always recover, effectively immunising them against the spread of the activity. In particular, we are interested in targeting graph properties that are highly skewed in the hyperbolic graphs such as high closeness, and high clustering. Nodes are selected a priori, based on the specified properties. Our intuition is that properties with highly skewed distributions will maximise the inhibitory effect. As a control, the simulations are repeated, selecting the same number of nodes at random to immunise in each simulation.

We theorise that it may be possible to determine, from graph properties, which set of nodes might have the most influence when effectively immunised against the bootstrap percolation process.

IV. Experimental Set Up

Our experimental design involves creating a set of hyperbolic graphs, simulating agent based modelling of bootstrap percolation on all graphs in this set, then simulating bootstrap percolation with immunity on selected graphs. Using the same set of graphs allows direct comparison across the different sets of simulations, and keeping the number of nodes fixed allows us to reduce complexity and present clearer results. All of our graphs have 1000 nodes as this is sufficiently large to analyse complex contagion, yet small enough to be computationally tractable. This synthetic graph model has been chosen as it is easier to create and control the parameters than in real world networks. This particular model has been chosen as it shares many features with real world complex networks, such as high clustering, short path lengths and power law degree distribution.

A. Creation of hyperbolic random geometric graphs

A set of hyperbolic random geometric graphs is created using the method outlined in Krioukov et al. [30]. This involves taking a disc of radius R, distributing points with a hyperbolically uniform node density and connecting points if the hyperbolic distance between them is less than R, with R varying from 0 to 12 in increments of 0.1.

Algorithm 1 Create Hyperbolic Geometric Graph $G(n,R)$ in a disc of hyperbolic radius R in $H2$

Input: Number of vertices n, distance parameter $R = 2log(n) + c$
Output: Adjacency matrix: $A[i,j]$
For each vertex $v_i \in V[G]$
for $i \leftarrow 1$ to n **do**
  Generate hyperbolically distributed independent polar co-ordinates:
  $r \in [0,R]$, $\theta \in [0,2\pi]$
  For each pair of vertices $v_i, v_j \in V[G]$
  for $i \leftarrow 1$ to n **do**
    for $j \leftarrow 1$ to $n-1$ **do**
      Calculate hyperbolic distance $d[v_i - v_j]$ [1], [2]
      if $d < R$ **then**
        $A[i,j] \leftarrow 1$
      else
        $A[i,j] \leftarrow 0$

For ease of comparison, all graphs are created with 1000 nodes, and 20 graphs are created at each distance parameter R, resulting in the creation of a set of 2,400 hyperbolic random geometric graphs with increasing edge density from 0 to 1.

B. Simulation of Bootstrap Percolation

For each graph, a population of agents is attached to each graph, one to each node. All agents are initially inactive, and 20 agents are chosen at random to form the active seed set A_o, representing a small scale random attack in a network. The initial activation threshold is set at 2 and, at each time step, inactive nodes with at least this number of active neighbours are activated. This mechanism is repeated for each time step until an equilibrium is reached, where no further state change is possible. The simulations are repeated on this graph increasing the activation threshold up to 10, in increments of 1. This is repeated for 1000 simulations at each activation threshold parameter, each with a different randomly selected seed set A_o. The final number of active nodes at equilibrium is recorded, and the outcome for each simulation is grouped in sets of one hundred, from 0 - 100 up to 901 - 1000; the number of simulations containing outcomes within each grouping is recorded.

In the previous heat map, Figure 3, it can be seen that for most graphs in the set, the activity either percolates completely or fails to percolate, we are particularly interested in the set of graphs at the percolation threshold. It is here that allowing for immunity is likely to have greater impact. For this reason we have chosen particular sets of graphs at the threshold to repeat our bootstrap percolation simulations, allowing for the introduction of immunised nodes in these graphs.

Our simulations for comparison of bootstrap percolation with bootstrap with immunity are therefore performed on 20 hyperbolic random geometric graphs created at each of two distance parameters R, specifically those at both edges of the upper boundary of the percolation threshold shown in figure 3. At the rightmost edge, $R = 5.7$, all simulations on this set of graphs completely percolated for all activation thresholds from 2 to 10. At the left edge, graphs created at distance parameter $R = 3.0$ were the first set which displayed complete percolation at activation threshold 2. These represent edge densities of approximately 0.1 and 0.02 respectively.

C. Simulation of Bootstrap Percolation with Immunisation

This set of experiments has the same design as the standard bootstrap percolation set up, however an immunised seed set I_o is selected beforehand to be immune to the percolation process, neither active nor inactive, and remaining immune throughout the simulation. Our initial experiments involved an immunised seed set of 20 immune nodes, the same size as the active seed set A_o. With 20 immune nodes we noted that the rate of decline in percolating simulations appeared linear, so we increased I_o to 25, and noted that it levelled off for values over 20 (see Figure 5). Therefore we chose I_o of 25 nodes as this gave a more complete picture of the decline in percolating simulations as the number of immune nodes increased.

1) Selection of Immune Nodes: Selection of I_o is performed in two ways, by targeted selection based on various node properties, and by random selection for comparison.

- Random selection
- Targeted selection
  - Degree centrality
  - Betweenness centrality
  - Closeness centrality
  - Watts Strogatz Clustering coefficient

Of the many options for centrality measures, we have chosen degree, betweenness and closeness as these have distinct and varying levels of skewed distribution in the hyperbolic graphs. This facilitates tracking the influence of each property to determine if there is any correlation between the level of skew of the centrality measure and the potential inhibitory effect of selecting nodes highly ranked for this property.

Degree centrality is the number of links incident to a node, with high degree representing a node with many contacts. Betweenness centrality measures the number of times a node appears on the shortest path between all pairs of nodes in the network, where high betweenness represents nodes which may

[1] Since $d[v_i - v_j] = d[v_j - v_i]$, then $A[i,j] = A[j,i]$
[2] Calculate hyperbolic distance $d[v_i - v_j]$ using: $Cosh(d) = cosh(r_i)cosh(r_j) - sinh(r_i)sinh(r_j)cos(\theta_i - \theta_j)$

have a “brokerage” role in a network. Closeness centrality is measured by the average length of the shortest path between the node and all other nodes in the graph, with high closeness representing nodes that are closely linked to important nodes while not necessarily having high degree or betweenness. Our selected clustering coefficient is the Watts Strogatz clustering coefficient, also known as the network average clustering coefficient. For an undirected graph the local clustering coefficient of a node is:

$$c_i = \frac{e_i}{\frac{k_i(k_i-1)}{2}}$$

where (e_i) is the number of connections in the neighbourhood of a node and $\frac{k_i(k_i-1)}{2}$ is the maximum number of connections in the neighbourhood of a node of degree k. The Watts Strogatz clustering coefficient is the average of the local clustering coefficients (c_i) over all nodes (n):

$$\bar{C} = \frac{1}{n}\sum_{i=1}^{n} c_i$$

All nodes in the graph are a priori ranked for the chosen property and the top 25 highest ranked nodes are selected for immunity. These nodes remain immunised and cannot change state. The experiments are repeated for 1000 simulations at each activation threshold from 2 up to 10.

V. Results

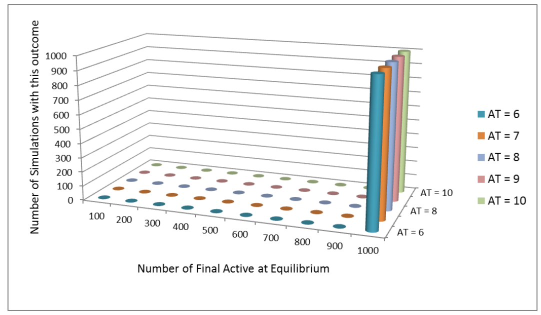
A. Upper boundary of percolation threshold for activation threshold of 10

The outcomes for all graphs at this parameter were similar. To clearly illustrate these outcomes we have charted the results from one particular graph randomly chosen at this parameter, $R = 5.7_{13}$. On this graph, we compare standard bootstrap percolation outcomes with outcomes from random selection of immunised nodes and then with targeted selection of immunised nodes. During a priori node selection for immunity, it was noted that the nodes of highest degree were also the same set of nodes top ranked for high closeness, high betweenness and low clustering. The results for high degree immunity therefore represent outcomes for these other node properties.

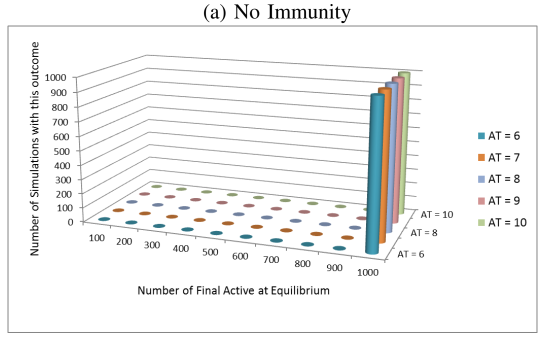
Figure 4 represents outcomes for 1000 simulations of bootstrap percolation at each activation threshold on one representative graph at the upper edge of the percolation threshold for activation threshold 10, the only thing varying in each simulation is the random selection of 20 active seeds in A_o. Each grouping along the x-axis represents the number of final active nodes at equilibrium. The height of each column represents the number of simulations that had outcomes within that grouping.

In the standard bootstrap percolation simulations, see figure 4a, all simulations completely percolated, confirming the initial results shown in the previous heat map.

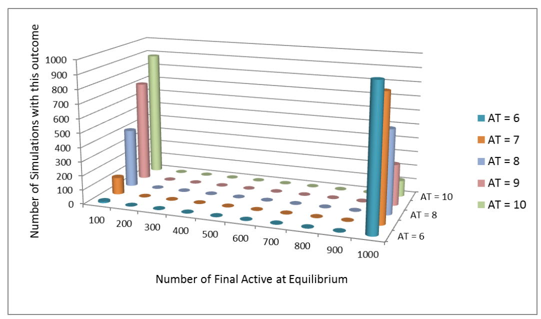
Figure 4b represents outcomes for randomly selected immune nodes showing that random immunity had no impact on delaying percolation, with all simulations completely percolating.



(a) No Immunity



(b) Random immunity



(c) Immunised for High Degree

Fig. 4: Graph 5.7_{13}. Comparison of outcomes for Immunity, with 1000 simulations at each activation threshold

Figure 4c shows outcomes for 1000 simulations with the top ranked 25 nodes of highest degree selected for immunisation. There was no impact on percolation for activation thresholds from 2 to 5, all simulations completely percolated, as before. However, it is clear that targeted selection of hub nodes has a significant impact on the percolation process for activation thresholds 6 to 10. In standard bootstrap percolation, all simulations at activation threshold 10 completely percolated. With targeted immunity, approximately 90% of the simulations now failed to percolate. This suggests that these 25 nodes selected for immunity are highly influential in the percolation process.

To assess if any particular node in I_o had a marked contribution to the delaying effect of immunity, we analysed the rate of percolation as the number of immunised nodes increased from 1 to 25 and found that no node had any outstanding contribution to the delaying effect on percolating simulations, see Figure 5. As the number of immunised nodes increased, the effect was cumulative with the rate of decline in percolating simulations following a logistic curve.

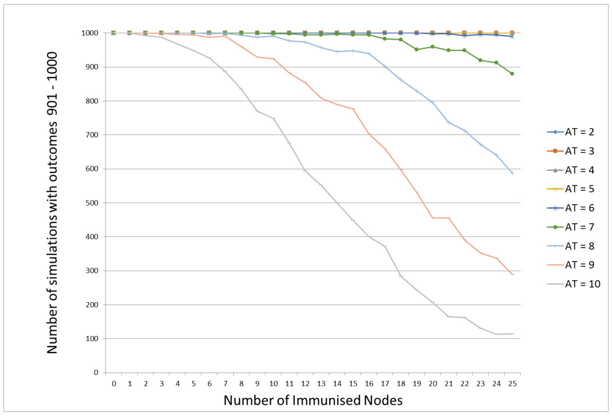


Fig. 5: Rate of decline in percolating simulations as number of immunised nodes increases from 0 to 25

B. Upper boundary of percolation threshold for activation threshold 2

All graphs at this edge density parameter had similar outcomes. For clarity, results from one representative graph at $R = 3.0_{13}$ are displayed in Figure 6.

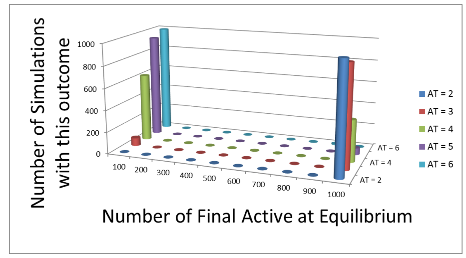
Additionally, results for activation thresholds of 7 to 10 are not included as all simulations at these parameters failed to percolate in standard bootstrap percolation, and immunity therefore could not have affected outcomes.

The results for simulations with Targeted Immunity on graphs at $R = 3.0$ are more varied than for graphs at $R = 5.7$. As before, random immunity had no impact when compared to outcomes from the standard bootstrap percolation process, see Figure 6b. However, when we selected the top ranked nodes for different graph properties for immunity, we had varying outcomes, see Figures 6c - 6f. In these plots, the greatest decline in percolation is most clearly seen by observing activation threshold 4. The delaying effect can be ranked as follows, with random immunity having no effect, when compared with standard bootstrap percolation; the greatest delaying effect was seen with nodes immunised for high closeness.

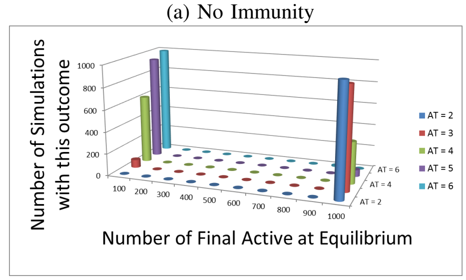
The rate of decline in percolating simulations is linear, with no immunised node having any particular contribution over any other immunised node.

- Ranked effect of node properties on delaying percolation, from highest to lowest:
  - High Closeness
  - Low Clustering (Local node clustering coefficient)
  - High Betweenness
  - High Degree
  - Random Immunity
  - Standard Bootstrap

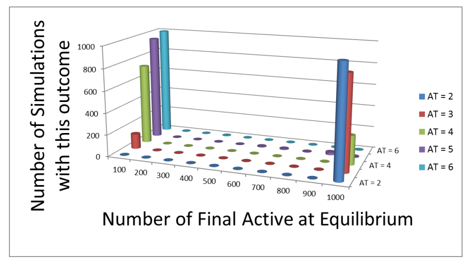
This is an interesting result, as hub nodes are commonly seen as the standard influential nodes in a network. This



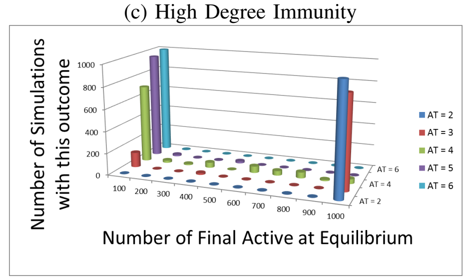
(a) No Immunity



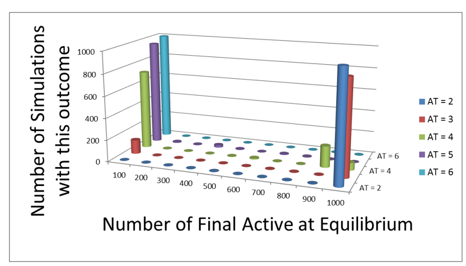
(b) Random Immunity



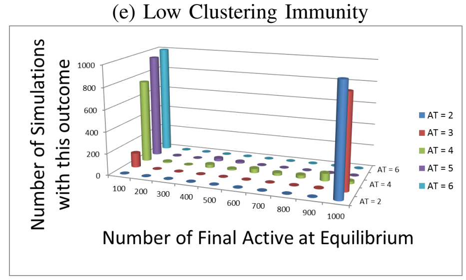
(c) High Degree Immunity



(d) High Betweenness Immunity



(e) Low Clustering Immunity



(f) High Closeness Immunity

Fig. 6: Graph 3.0_{13}. Comparison of outcomes for Immunity, with 1000 simulations at each activation threshold

prompted an analysis of global graph properties at the upper boundary of the percolation threshold.

C. Graph properties at the upper boundary of the percolation threshold, for activation thresholds of 2 and 10

Analysis of the representative graphs at the upper boundary of the percolation threshold are presented in Table I, the global graph properties for each were similar to the other graphs at their respective edge densities. Figure 7 shows our selected graph centralisation measures as the number of edges increases in the set of graphs, with selected threshold graph parameters marked as blue lines.

TABLE I: Global graph properties at the upper boundary of the percolation threshold

Graph Properties	Graph 3.0_{13}	Graph 5.7_{13}
Density	0.019416	0.098962
Average Degree	19.416	98.962
Diameter	7	3
Watts Strogatz Clustering	0.7306781	0.78016659
Transitivity	0.55326337	0.4753645
Degree Centralisation	0.03970303	0.63995659
Closeness Centralisation	0.33503071	0.60320727
Betweenness Centralisation	0.21560217	0.12449812

It is interesting to note that the network average clustering coefficient is very highly skewed in both graphs, which might suggest that nodes with high local node clustering would be influential in the percolation process. However, our results show that it is nodes of low local clustering values that have more impact than high clustering. This effect could, in part, be due to the nature of the hyperbolic graphs, which typically display central hub nodes (high degree) with leaf nodes towards the boundary. These hub nodes generally have low local node clustering values. The disparity of the number of edges in the hubs, compared with other nodes, means that transitivity, i.e. closing a triple, is less likely to occur. In fact the hyperbolic graphs display a marked difference in the values for the network average clustering coefficient and transitivity, with the latter significantly lower than the former.

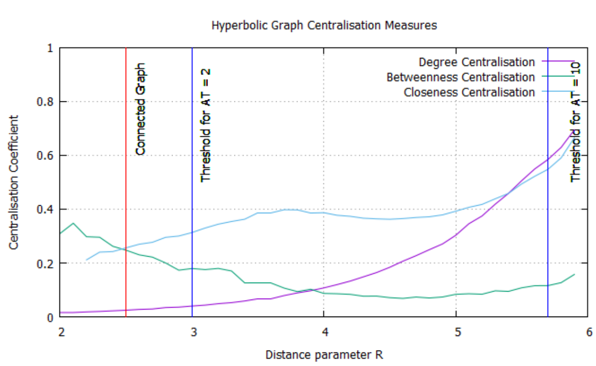


Fig. 7: Hyperbolic Graph Centralisation

D. Discussion

In graphs at the upper boundary of the percolation threshold for activation threshold 10 ($R = 5.7$), all simulations had previously percolated at all activation threshold values in standard bootstrap percolation. After targeting 25 nodes of high degree for immunity, this had a significant impact on delaying percolation, with 90% of simulations now failing to percolate at activation threshold 10. This set of 25 nodes also had the highest values for betweenness and closeness centrality, and for low clustering. These targeted measures matched our expectations based on global graph properties. However, for graphs at the upper boundary of the percolation threshold for activation threshold 2 ($R = 3.0$), the results were more ambiguous.

In graphs at $R = 3.0$, the impact on percolation was more varied, with nodes of high closeness having most impact, followed by high degree, low clustering, and high betweenness. This does not match our expectations based on the degree of skew seen in the graph properties. The greatest skew was seen in clustering coefficients, followed by closeness and then betweenness, with the lowest skew for degree centralisation. However, when we restrict our analysis to the chosen centrality measures, there was a good correlation between skewed centralisation and the impact on delayed percolation. Our work confirms previous research on networks that hub nodes are influential in spreading activities on networks, but also highlights the greater importance of other centrality measures on hyperbolic networks of varying edge density. It may be that some other measure, or a combination of properties, is at the heart of the influence and that it might be possible to fine tune influential node selection by observing a variety of graph properties.

VI. Conclusion

This work has demonstrated that identifying influential nodes and targeting them for immunity has an inhibitory effect on the bootstrap percolation process on hyperbolic networks, when compared with random immunity.

This indicates that in these graphs certain nodes are highly influential in the network and warrant being protected. Given a small scale attack in a network, with 20 active seeds, our results suggest that under these conditions it is possible to immunise influential nodes and effectively grant herd immunity to the whole network. In the case of the graphs at activation threshold 10, by targeting 25 nodes with high centrality measures, we have reduced the likelihood of percolation from 1 to 0.1. On graphs at edge density 0.2, the probability of percolation for activation threshold 4 has decreased from 0.4 to 0.02, for activation threshold 3, the probability of percolation has decreased from 0.97 to 0.86. On this set of graphs, the rate of decline was linear, which suggests that increasing the number of immunised nodes would increase the obstructive effect on percolation.

Results from our simulations have demonstrated that targeting immunity at the top ranked nodes for high degree, closeness and betweenness centrality and for low local node clustering coefficients has a delaying impact on the dynamics of the bootstrap percolation process on the set of hyperbolic graphs, when compared with random immunity which had no effect on the bootstrap process. This impact varies with the choice of node property selected. Node degree is commonly seen as an important indicator of influence in a network and our results confirm this, however in our graphs at the percolation threshold for activation threshold 2, closeness centralisation has the most effect in impeding the spread of the activity. In terms of global graph properties, the selected centrality measures had a good correlation with the degree of skew on their respective graph centralisation measures, with the top ranked nodes of highest skew relating to a greater impact on delaying percolation.

1) Future work: An interesting avenue of follow-up research is to replicate our work on all of the hyperbolic graphs within the percolation threshold zone, with a view to mapping the relationship with edge density, the changing pattern of graph centralisation measures and outcomes in the percolation process. The threshold graphs are chosen as it is here that any potential delaying impact can be readily observed. We are interested in those simulations which had a marked change in our immunity experiments, to examine the relationship between the active seed set and the immunised nodes. In our simulations at the percolation threshold for activation threshold 10, $R = 5.7$, only 10% of simulations completely percolated after the introduction of immunity. By investigating the spatial configuration of the paths between these sets of nodes, and the local neighbourhood structure, it might be possible to determine which local neighbourhood features have the greatest impact on global dynamics in the network.

We plan to use a greater variety of global graph measures to predict which nodes properties are influential in a dynamic process. This would require examining a greater subset of the hyperbolic graphs to see if our predicted properties match simulated outcomes.

Acknowledgments

This research was supported by the Irish Research Council Postgraduate (Government of Ireland) funding.

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